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Theory of Local Heat Transfer in Shock/Laminar Boundary-Layer Interactions

George R. Inger*
Iowa State University, Ames, Iowa 50011-3231

A fundamental theory of the local heat transfer disturbances in supersonic and hypersonic shock/laminar boundary-layer interaction zones is given, based on an asymptotic nonadiabatic triple-deck model that is formulated using the reference temperature method combined with a total enthalpy form of the energy equation. Analytical and numerical results are presented and compared with experimental heat transfer data for compression corner-generated interactions. The predictions of the theory provide strong support for important but empirical observations on the behavior of interactive heat transfer when separation occurs.

Nomenclature

C	=	Chapman–Rubesin parameter, $\mu T_\infty/\mu_\infty T$
C_f	=	skin friction coefficient, $2\tau_W/\rho_\infty U_\infty^2$
C_p	=	specific heat
H	=	total enthalpy, $C_pT + u^2/2$
K	=	incipient separation correlation constant, Eq. (31)
K_H	=	$(\gamma + 1)\lambda^{1/2}M_\infty^2 C_{\mathrm{REF}}^{1/4}\varepsilon^2/4\beta^{1/2}$
L	=	reference length, Fig. 1
M	=	Mach number
Pr	=	Prandtl number
p	=	static pressure
q_w	=	wall heat transfer rate
Re_L	=	Reynolds number $\equiv \varepsilon^{-8}$, $\rho_\infty U_\infty L/\mu_\infty$
s_e	=	total streamline slope along the boundary-layer edge, $v_e/U_\infty + \theta_B$
T	=	absolute static temperature
T_t	=	freestream total temperature, H_{0_∞}/C_p
U_∞	=	freestream velocity at edge of incoming boundary layer
u, v	=	velocity components in x, y directions, respectively
x, y	=	streamwise and normal coordinates, respectively
β	=	$(M_\infty^2 - 1)^{1/2}$
γ	=	specific heat ratio
δ^*	=	displacement thickness variable
δ_0^*	=	undisturbed boundary-layer displacement thickness
Θ	=	flow deflection angle
κ	=	$K/\lambda^{1/2}\hat{\theta}_{\mathrm{is}}$
λ	=	0.332, Blasius solution constant
μ	=	coefficient of viscosity $\equiv\rho\upsilon$
ρ	=	density
τ_w	=	wall shear stress
$\tilde{\chi}$	=	viscous interaction parameter, $C_{\mathrm{REF}}M_\infty^3 Re_L^{-1/2}$
$\tilde{\tilde{\chi}}$	=	unified supersonic–hypersonic interaction parameter, $M_\infty\tilde{\chi}/\beta$
ω	=	viscosity temperature-dependence exponent, $\mu \sim T^\omega$

Subscripts

ADIAB	=	adiabatic wall conditions
B	=	body surface
e	=	local inviscid flow conditions at boundary-layer edge
i.s.	=	incipient separation
REF	=	based on reference temperature
w	=	wall surface conditions
0	=	undisturbed boundary layer ahead of interaction zone
∞	=	freestream conditions

Superscript

^	=	nondimensional variables from triple-deck theory, Eqs. (12–19)

I. Introduction

INTERACTIONS between oblique shock waves and boundary layers must be understood to predict the performance of aerodynamic devices such as flaps, spoilers, and inlets. These involve strong viscous/inviscid interaction flow with a large adverse pressure gradient that often provokes boundary-layer separation. The prediction of the onset of such separation and the delineation of the underlying scaling laws that govern it continue to be important in aerodynamic studies of high-speed aircraft and missiles; these vehicles operate and are tested over a wide range of Mach and Reynolds numbers. This paper re-examines the fundamental similitude rules pertaining to the laminar (high-altitude) flight regime, with the goal of establishing a single unified scaling law for the incipient separation pressure from supersonic to strongly hypersonic Mach numbers on nonadiabatic surfaces.

Knowledge of the corresponding heat transfer disturbances in such interaction zones is also of concern because of their importance in the aerothermodynamic design of cooled hypersonic flight vehicles and because such heat transfer is itself an important diagnostic in understanding the interactive flow and its separation. Although a significant experimental database on shock/boundary-layer interaction (SBLI) zone heat transfer has accumulated, along with purely computational fluid dynamics (CFD)-type predictions, comparatively little has been done on the basic theory of the problem. Thus, while asymptotic (triple-deck) methods have been fruitfully and extensively applied to analyze all aspects of the local pressure and skin friction behavior under adiabatic wall conditions[1–3] and to establish the similarity rules governing incipient separation in both supersonic and hypersonic nonadiabatic flows,[4,5] only two investigations have addressed the interactive heat transfer disturbance aspect per se. One of these dealt strictly with subsonic flows

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*Professor, Department of Aerospace Engineering. Associate Fellow AIAA.

around small bumps or troughs[6]; the other, which forms the basis for the work addressed here, treated the case of moderately supersonic interactions with compression corners.[7] The second goal of the present study is to reformulate and extend the latter analysis to include the hypersonic regime, and then to give comparisons of the resulting interactive heat transfer predictions with experimental data. The results may be helpful in understanding purely numerical Navier–Stokes solutions of the problem that require very fine computational meshes and large CPU times to resolve the surface heat transfer aspects.

II. Analysis Formulation

A. Assumptions and Governing Equations

We consider two-dimensional steady laminar flow of an ideal gas in the strong local interaction region associated with either an impinging shock or a compression corner. We specifically treat compression corner interactions (Fig. 1); the companion impinging shock problem follows similar lines.[8] Our method of approach is the so-called triple-deck theory in its leading high-Reynolds-number approximation, recast into a form suited to nonadiabatic flows by means of the reference temperature concept combined with total enthalpy formulation of the energy equation. We note in this regard that whereas Ref. 7 uses the temperature form of the energy equation, we here employ a total enthalpy formulation instead because of the latter's well-known superior ability to account for the heat transfer aspects of a viscous flow when the Mach number is strongly hypersonic.[9] We shall also address the influence of large wall cooling on the hypersonic interactive displacement effect, which was not considered in Ref. 7.

We deal here with the situation where upstream global viscous–inviscid interaction or nose bluntness effects ahead of the corner may be neglected. The disturbance flow physics within a short-ranged viscous–inviscid interaction zone may then be organized into three distinct decks when the Reynolds number is high[1,3] (Fig. 1): 1) An outer layer external to the boundary layer consisting of inviscid disturbance flow associated with the viscous displacement effect of the underlying decks, 2) a middle layer containing rotational particle-isentropic nonadiabatic disturbance flow dependent on the boundary-layer profile, and 3) a thin inner deck of nonadiabatic viscous-disturbance flow within the linear portion of this profile that is interactively coupled with the local pressure field. In terms of the basic small perturbation parameter $\varepsilon \equiv Re_L^{-1/8}$, these decks have thicknesses of the order $\varepsilon^3 L$, $\varepsilon^4 L$, and $\varepsilon^5 L$, respectively, along a streamwise interaction zone length of order $\varepsilon^3 L$. Taken in its leading asymptotic approximation as $\varepsilon \to 0$ (very high Reynolds number) this theory yields the following set of governing disturbance flow equations when the external Mach number ranges from supersonic to hypersonic: In the middle deck we have

$$\frac{v(x, y)}{U_0(y)} \cong \frac{v_e}{U_{0e}} - \frac{\dfrac{\mathrm{d}p}{\mathrm{d}x}}{\gamma p_0} \int_y^{\delta_0} (M_0^{-2} - 1)\, \mathrm{d}y \tag{1}$$

$$u \cong U_0(y) - \frac{\mathrm{d}U_0}{\mathrm{d}y}(y_B + \delta^*) \tag{2}$$

$$H \cong H_0(y) - \frac{\mathrm{d}H_0}{\mathrm{d}y}(y_B + \delta^*) \tag{3}$$

where $U_0(y)$ and $H_0(y) = C_p T_0 + U_0^2/2$ are the velocity and total enthalpy profiles of the incoming nonadiabatic undisturbed boundary layer, whereas δ^* is an unknown displacement thickness that is linked to the interactive pressure perturbation as shown later in this paper. We note here that the term involving the Mach-number profile integral in Eq. (1), while dropped in the standard triple deck formulation,[3] is retained at this point because it carries a potentially significant effect of large wall cooling in strongly hypersonic flow. Within the underlying very thin inner deck, the viscous, heat-conducting disturbance flow is governed by the following equations:

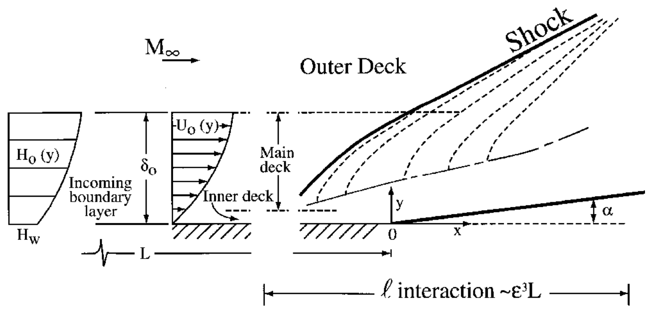

Fig. 1 Triple deck and flow structure for a compression ramp-generated interaction region.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \rho_w^{-1} \cdot \left[1 + \left(\frac{H_\infty}{Hw} - 1\right)\frac{U_0}{U_\infty}\right]\frac{\mathrm{d}p}{\mathrm{d}x} = \upsilon_w \frac{\partial^2 u}{\partial y^2} \tag{5}$$

$$u\frac{\partial H}{\partial X} + v\frac{\partial H}{\partial y} \cong \frac{\upsilon_w}{Pr}\frac{\partial^2 H}{\partial y^2} \tag{6}$$

where a term proportional to $(U_0/U_\infty)^2$ has been neglected compared to unity in the square-bracket coefficient of Eq. (5), whereas in Eq. (6) a term proportional to $(1 - Pr)\partial/\partial y(u\partial u/\partial y)$ has been neglected because of the small velocities in the inner deck (this is a well-known satisfactory simplification even in very strongly hypersonic flows[9]). We also note that the values of ρ_w and υ_w pertain to the prevailing highly cooled wall temperature. The solutions of Eqs. (5) and (6) are subject to the no slip/impermeable wall boundary conditions $u(x, y_B) = v(x, y_B) = 0$ on the body surface ($y_B = 0$, $x \le 0$; $y_B = \varsigma_B x$, $x > 0$), where on this surface $H(x, y_B) = H_w = C_p T_w$ is given with the corresponding heat transfer to be determined as

$$-\dot{q}_w = \frac{\mu_w}{Pr}\frac{\partial H}{\partial y}(x, y_B) \tag{7}$$

The matching at the aforementioned middle and inner decks is achieved to the leading asymptotic approximation by enforcing the relationships that

$$u_{\text{inner}}(x, y \to \infty) = \frac{\tau_{w_0}}{\mu_{w_0}}(y - y_B - \delta^*) \tag{8}$$

$$H_{\text{inner}}(x, y \to \infty) = \frac{Pr(-\dot{q}_{w0})}{\mu_{w0}}(y - y_B - \delta^*) \tag{9}$$

where the undisturbed values of the wall shear τ_{w0} and heat transfer q_{w0} will be evaluated on the basis of the reference temperature method as shown next. We also match the vertical velocity value obtained from the inviscid limit of Eq. (5) as $y \to \infty$ with the middle deck value given by the $y \to 0$ limit of Eq. (1). Taking note of Eq. (7), inspection of Eqs. (3), (6), and (9) immediately shows that local interactive heat transfer disturbances will occur only if the incoming boundary layer is also nonadiabatic. We re-emphasize that according to the leading asymptotic approximation, Eqs. (1–9) are valid for all values of M_∞ and $T_w/T_{w,\text{ADIAB}}$ including the presence of wall heat transfer.

B. Viscous–Inviscid Interaction Relationship

The forgoing equations must be supplemented by a further relationship between the interactive pressure and displacement thickness δ^*, and it is here that the analysis becomes specialized to the Mach-number regime and streamwise scale adopted for study. In the present work we are concerned with the local events in the immediate vicinity of the compression corner (or shock impingement point) to the exclusion of any larger-scale effects because of a blunt nose or leading edge. We thus adopt the p_e vs s_e relationship given by the tangent wedge approximation, which is known to be accurate throughout the combined supersonic–hypersonic flow regimes[9]:

$$\frac{p - p_\infty}{\gamma p_\infty M_\infty^2} \cong \left[\frac{M_\infty s_e}{\beta}\right]^2 \left\{ \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \left[\frac{\beta}{M_\infty^2 s_e}\right]^2} + \left(\frac{\gamma+1}{4}\right)\right\} \tag{10}$$

In the limit of weak interactions characterized by $M_\infty^2 s_e << \beta$ this reduces to the linearized supersonic flow value s_e/β.

C. Rescaling the Problem Using the Reference Temperature Method

We now reformulate this triple-deck model using the reference temperature method, because it has been shown to perform well in computational predictions of hypersonic compression ramp interaction heating[10] and has also provided the nonadiabatic wall effect on the incipient separation criteria for such flows.[4,5] According to this method the combined effects of compressibility and heat transfer on the incoming boundary layer can be accurately accounted for by evaluating its physical properties at a reference temperature T_{REF} that depends on γ, M_∞ and $T_w/T_{w,\mathrm{ADIAB}}$. Adopting Eckerts result, which has been shown to have a fundamental basis in compressible boundary-layer theory,[11] this gives for air with $Pr = 0.72$ that

$$T_{\mathrm{REF}}/T_\infty \cong 0.50 + 0.039M_\infty^2 + 0.50(T_w/T_\infty) \tag{11}$$

We take $\mu \sim T^\omega$ with $0.50 \le \omega \le 1$ and use $(\mathrm{d}U_0/\mathrm{d}y)_w = \tau_{w_0}/\mu_{w_0} = \lambda\varepsilon^4(C_{\mathrm{REF}}^{1/2}T_{\mathrm{REF}}/T_\infty)^{-1}(T_{\mathrm{REF}}/T_w)^\omega \rho_\infty U_\infty^2/\mu_\infty$ plus $-\dot{q}_{w_0} = \tau_{w_0}(H_{\mathrm{ADIAB}} - C_pT_w)/U_\infty Pr^{2/3}$ (Ref. 11). We now introduce the nondimensional rescaled variables

$$\hat{x} \equiv [x/L]\lambda^{5/4}\kappa^{5/2}\beta^{3/4}/\varepsilon^3 C_{\mathrm{REF}}^{3/8}(T_{\mathrm{REF}}/T_\infty)^{3/2}(T_w/T_{\mathrm{REF}})^{\omega+1/2} \tag{12}$$

$$\hat{y} \equiv (y/L)\lambda^{3/4}\kappa^{3/2}\beta^{1/4}/\varepsilon^5 C_{\mathrm{REF}}^{5/8}(T_{\mathrm{REF}}/T_\infty)^{3/2}(T_w/T_{\mathrm{REF}})^{\omega+1/2} \tag{13}$$

$$\hat{u} \equiv (u/U_\infty)\beta^{1/4}/\kappa^{1/2}\varepsilon C_{\mathrm{REF}}^{1/8}\lambda^{1/4}(T_w/T_\infty)^{\omega+1/2} \tag{14}$$

$$\hat{v} \equiv (v/U_\infty)/\kappa^{3/2}\varepsilon^3\beta^{1/4}C_{\mathrm{REF}}^{3/8}\lambda^{3/4}(T_w/T_\infty)^{1/2} \tag{15}$$

$$\hat{p} \equiv \lfloor (p - p_\infty)/\rho_\infty U_\infty^2 \rfloor \beta^{1/2}/\varepsilon^2 C_{\mathrm{REF}}^{1/4}\lambda^{1/2}\kappa \tag{16}$$

$$\hat{H} \equiv \beta^{1/4}(H - C_pT_w)/Pr^{1/3}(H_{\mathrm{ADIAB}} - C_pT_w) \times \kappa^{1/2}\varepsilon C_{\mathrm{REF}}^{1/8}\lambda^{1/4}(T_w/T_\infty)^{1/2} \tag{17}$$

$$\hat{\theta} \equiv \theta/\kappa\varepsilon^2\lambda^{1/2}\beta^{1/2}C_{\mathrm{REF}}^{1/4} \tag{18}$$

$$\hat{\delta} \equiv (\delta^*/L)\lambda^{3/4}\kappa^{3/2}\beta^{1/4}/\varepsilon^5 C_{\mathrm{REF}}^{3/8}(T_{\mathrm{REF}}/T_\infty)^{3/2}(T_w/T_{\mathrm{REF}})^{\omega+1/2} \tag{19}$$

where $\lambda = 0.332$ and $C_{\mathrm{REF}} \equiv \mu_{\mathrm{REF}}T_\infty/T_{\mathrm{REF}}\mu_\infty$ are the Blasius and Chapman–Rubesin constants, respectively. Note that this scaling is a unified one that pertains to both supersonic and hypersonic flow situations, the latter limit pertaining to the $M_\infty^2 >> 1$ limit of Eq. (10). Then after some detailed algebra, Eqs. (1–9) plus the vertical velocity matching yield the following set of universal nondimensional equations governing the present nonadiabatic interaction problem:

$$\hat{s}_e \equiv \hat{v}_e + \hat{\theta}_B = \frac{\mathrm{d}(\hat{\delta}^* + \hat{y}_B)}{\mathrm{d}\bar{x}} + s_I I \frac{\mathrm{d}\hat{p}}{\mathrm{d}\hat{x}} \tag{20}$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{21}$$

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\mathrm{d}\hat{p}}{\mathrm{d}\hat{x}}(1 + \varepsilon K_u\hat{y}) = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \tag{22}$$

$$\hat{u}\frac{\partial \hat{H}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{H}}{\partial \hat{y}} = Pr^{-1}\frac{\partial^2 \hat{H}}{\partial \hat{y}^2} \tag{23}$$

$$\hat{u}(\hat{x}, \hat{y} \to \infty) = \hat{y} - \hat{y}_B - \hat{\delta}^* \tag{24}$$

$$\hat{H}(\hat{x}, \hat{y} \to \infty) = \hat{y} - \hat{y}_B - \hat{\delta}^* \tag{25}$$

where

$$K_u \equiv \lambda^{1/4}C_R^{1/8}(T_w/T_\infty)^{1/2}\beta^{-1/4}[(T_t/T_w) - 1] \tag{26}$$

$$S_I = \varepsilon M_\infty^2\lambda^{5/4}/\beta^{1/4}C_R^{3/8}(T_{\mathrm{REF}}/T_\infty)^{1-\omega}(T_w/T_\infty)^{\omega+1/2} \tag{27}$$

$$I \equiv \int_{\hat{y}_{LS}}^{\delta^*}(M_0^{-2} - 1)\,\mathrm{d}\hat{y} + \hat{y}_{LS} - \{\hat{y}_{LS}[M_0'(0)]^2\}^{-1} \tag{28}$$

is a nondimensional boundary-layer Mach-number profile integral that is of order unity. These equations are to be solved subject to the boundary conditions $\hat{u} = \hat{v} = \hat{H} = 0$ along $\hat{y} = 0$ for $\hat{x} < 0$ and $\hat{y} = \hat{y}_B = \theta_B\hat{x}$ for $\hat{x} \ge 0$. Similarly, Eq. (10) yields the nondimensional pressure relationship

$$\hat{p} = \hat{s}_e[\sqrt{1 + (K_H\hat{s}_e)^2} + K_H\hat{s}_e] \tag{29}$$

where $K_H \equiv (\gamma + 1)\lambda^{1/2}M_\infty^2 C_{\mathrm{REF}}^{1/4}\varepsilon^2/4\beta^{1/2}$ is a new local viscous interaction parameter that is proportional to $\tilde{\chi}^{1/2}$. We note that Eq. (29) generalizes the $\hat{p} - \hat{s}_e$ relationship of classical triple-deck theory to the case of arbitrary hypersonic Mach numbers; in the limit $K_H \to 0$, it passes over to the usual linearized supersonic result $\hat{p} = \hat{s}_e$. Again, the foregoing equations apply to short-ranged local interactions at all M_∞ and $T_w/T_{w,\mathrm{ADIAB}}$ values, within the leading order of asymptotic high-Reynolds-number approximation and reference temperature concept adopted.

At this point it is possible to simplify the foregoing equations in two significant ways for the practical applications envisioned by the present study. First, because the effective height of the inner deck (units of $\hat{y}$) is very small, we find that the term $\varepsilon K_u\hat{y}$ in Eq. (22) can still be neglected compared to unity even for rather cold walls; although $H_\infty/h_w >> 1$ for such walls, an estimate of $\hat{y}_{\max}$ based on the inner deck theory of Lighthill[12] shows that $\varepsilon K_u\hat{y}_{\max}$ is always small provided h_w/H_∞ is above the hypersonic threshold value of $4\tilde{\chi}^{1/2}(T_t/T_\infty)/M_\infty^2(T_R/T_\infty)^{\omega+1}$, which is a very small value indeed (0.10 or less), unless the interaction parameter $\tilde{\chi}$ is unrealistically large compared to unity. Second, we find (Appendix A) that for the highly cooled wall conditions encountered on practical hypersonic aerodynamic vehicles and test facility models, the values of h_w/H_∞ involved (typically ≥ 0.1) are still large enough to yield negligible values of the hypersonic interaction term $S_I I(\mathrm{d}\hat{p}/\mathrm{d}\hat{x})$ compared with the other terms in Eq. (20), unless (again) the interaction parameter $\tilde{\chi}$ is very large compared to unity. Numerical studies of the effect of this term on the solution of Eqs. (21) and (22) confirm it to be small when $S_I \cdot I$ is small.[13]

For the foregoing reasons, we shall hereafter neglect the terms involving S_I and $\varepsilon K_u\hat{y}$ in Eqs. (20) and (22), respectively. This in turn reduces the resulting triple-deck Eqs. (20–25) to a form that has already been studied extensively. We now examine the consequent similitude laws governing the local interactive pressure and heat transfer.

III. Interactive Pressure Field

A. Upstream Influence

The distance x_{up} that the interaction extends upstream of the corner is an important physical property of the aerodynamic effect of the interaction. The nondimensional scale of this upstream influence is obviously a multiple of the coordinate $\hat{x}$; specifically, just prior to separation, numerical solution[1,3] of the foregoing triple-deck problem has shown that the upstream pressure decays like $\exp(\hat{x}/1.209)$ for $\hat{x} < 0$, so that we may define $\hat{x}_{up} = 1.209$. Then conversion back to physical variables from Eq. (7) yields

$$x_{up}/L = 1.209(T_R/T_0)^{1-\omega}\varepsilon^3 C_{REF}^{3/8}(T_0/T_\infty)^{3/2}(T_w T_0)^{\omega+1/2}/\beta^{3/4}\lambda^{5/4}\kappa^{5/2} \tag{30a}$$

or

$$(x_{up}/L)[M_\infty^2/(T_0/T_\infty)]^{3/2} = 1.209\lambda^{-5/4}\kappa^{-5/2}(T_R/T_0)^{1-\omega} \times (T_w/T_0)^{\omega+1/2}[(M_\infty/\beta)\tilde{\chi}]^{3/4} \tag{30b}$$

These relationships, which extend the earlier works of Stewartson[1] and Lighthill,[12] bring out several significant physical features of the upstream influence. First, it clearly shows that wall cooling ($T_w << T_t$) significantly reduces the streamwise extent of the interaction compared to the adiabatic wall case. This prediction is well supported by the available experimental evidence on hypersonic compressive interactions. Figure 2 shows an example from the well-known $M_\infty = 4$–6 study of Lewis et al.,[14] where incorporation of this cooling effect on the nondimensional x scaling of the interactive pressure data in the manner of Eq. (30) correlates both cooled ($T_w \cong 0.24T_t$) and adiabatic wall data on virtually one curve. Second, Eq. (30b) indicates that x_{up}/L in strongly hypersonic flow ($\beta \cong M_\infty$) scales as $\tilde{\chi}^{3/4}$, whereas in the more general supersonic-to-moderately hypersonic case it shows that the appropriate parameter is more generally the unified interaction parameter $\tilde{\tilde{\chi}} = M_\infty\tilde{\chi}/\beta$, which, of course, passes over to $\tilde{\chi}$ in the similitude governing the other features of the interaction to follow. Third, Eqs. (30) imply that the upstream influence decreases significantly with increasing Mach number, being proportional to $M_\infty^{-3/4}$ for $\omega = 1$ at hypersonic speeds.

B. Incipient Separation Criteria

The analytical and numerical solution of the triple-deck Eqs. (20–29) has been studied extensively.[1,3,13] These studies show that over a wide range of $\tilde{\chi}$ and $T_w T_{w,\mathrm{ADIAB}}$, the local wall shear stress vanishes slightly upstream of the corner at approximately the same value of the nondimensional deflection angle, namely, $\hat{\theta}_{i.s.} \cong 1.57$. That is, $\hat{\theta}_{i.s.}$ is virtually independent of the parameter K_H even though the streamwise location $x_{i.s.}$ itself may depend on K_H. This conclusion is verified a posteriori by agreement of its subsequent predictions with experiment as shown later in this paper. It is further supported by a detailed analysis of the inner deck vertical velocity field (see Appendix B) and by the general similitude analysis of hypersonic laminar-free interactions given in Ref. 2. The desired scaling law governing $M_\infty\theta_{i.s.}$ is then obtained as

$$(M_\infty/\beta)M_\infty\theta_{i.s.} = 1.57\lambda^{1/2}\kappa(\tilde{\tilde{\chi}})^{1/2} = K(\tilde{\tilde{\chi}})^{1/2} \tag{31}$$

which again involves the new unified interaction parameter $\tilde{\tilde{\chi}}$ (the appearance of the factor M_∞/β involved might have been anticipated on the basis of Van Dyke's[15] unified supersonic–hypersonic similitude rule governing the inviscid portion of the flow). In the strongly hypersonic limit $\beta \to M_\infty >> 1$, Eq. (31) passes over to the relationship $M_\infty\theta_{i.s.} = K(\tilde{\chi})^{1/2}$ with $K \simeq 1.4$–1.5 that has been found[16] to successfully correlate a large body of experimental data over a wide range of Mach and Reynolds numbers for both weak ($\tilde{\chi} < 1$) and strong ($\tilde{\chi} > 1$) interactions as discussed in Ref. 4. Further support of Eq. (31) from numerical computations has also been recently found in high-resolution Navier–Stokes solutions.[17] We note that Eq. (31) further implies $\theta_{i.s.} \approx \lfloor(M_\infty^2 - 1)/Re_L\rfloor^{1/4}$, i.e., a slow reduction of $\theta_{i.s.}$ with either increasing Re_L or decreasing M_∞. As regards the wall temperature effect, it is contained entirely in the parameter C_{REF} to the present approximation (see Ref. 4 for further elaboration on this aspect of the similitude).

One may alternatively express the foregoing incipient separation criterion in terms of pressure rise. To do this we first note that the interactive displacement thickness slope is necessarily proportional to the compression corner angle and, hence, upstream that $\tilde{s}_e = K_\delta\hat{\theta}_{i.s.}$, where K_δ remains essentially constant (≈ 0.7–0.8) over a wide range of $\tilde{\chi}$ values. Then combining this with Eqs. (18) and (21) and converting back to physical variables yields the following result regardless of the physical location $x_{i.s.}$:

$$\frac{p_{i.s.} - p_\infty}{\gamma p_\infty} \cong C_4\tilde{\chi}^{1/2}\cdot\left\{\sqrt{1 + \left[\left(\frac{\gamma+1}{4}\right)C_4\tilde{\chi}^{1/2}\right]^2} + \left(\frac{\gamma+1}{4}\right)C_4\tilde{\chi}^{1/2}\right\} \tag{32}$$

where $C_4 \equiv 1.57K_\delta$. In the weak interaction limit at moderate supersonic speeds where $\tilde{\chi} << 1$, Eq. (32) predicts that

$$\frac{p_{i.s.} - p_\infty}{\gamma p_\infty} \cong C_4\tilde{\chi}^{1/2} \tag{33a}$$

or

$$C_{p_{i.s.}} \sim (C_{REF}/Re_L)^{1/4}(M_\infty^2 - 1)^{-1/4} \tag{33b}$$

which is in fact a well-known scaling law for the free interaction zone of both ramp and impinging shock-generated laminar separating flows at supersonic and moderately hypersonic speeds.[1,2] Additional discussion of this limit and its further extension down to transonic speeds can be found in Ref. 4.

On the other hand, for strong interactions at hypersonic speeds with $\tilde{\chi} >> 1$. Eq. (32) predicts the separation pressure scaling

$$\frac{p_{i.s.} - p_\infty}{\gamma p_\infty} \cong C_4^2\left(\frac{\gamma+1}{2}\right)\tilde{\chi} \tag{34}$$

which exhibits a stronger ($\sim\tilde{\chi}$) dependence on $\tilde{\chi}$ than the weak

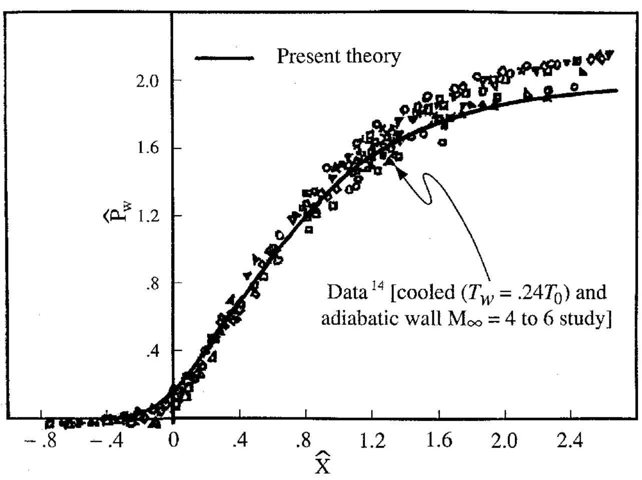

Fig. 2 Universal correlation of wall pressure distributions for nonadiabatic interactions.

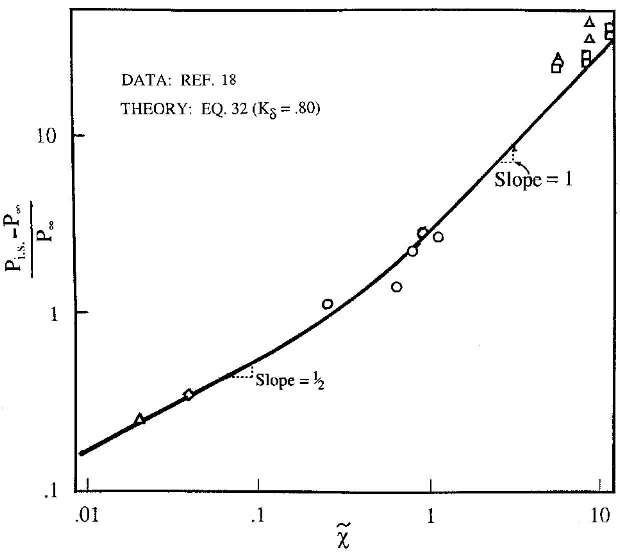

Fig. 3 Predicted pressure rise for incipient separation and comparison with experiment.

interaction result of Eq. (33). These predictions are well corroborated by a large body of experimental data,[18] as shown in Fig. 3: At small $\tilde{\chi} << 1$ the observed incipient separation pressures scale as $\tilde{\chi}^{1/2}$, whereas at larger $\tilde{\chi} \geq 1$ they clearly follow the linear dependence on $\tilde{\chi}$ indicated by Eq. (34). This predicted switchover behavior in the $\tilde{\chi}$ dependence of incipient separation pressure, which up to now has been observed but unexplained, is a significant result of the present work; it is now seen to follow from the leading asymptotic local approximation of triple-deck theory combined with the tangent wedge approximation for the outer inviscid flow. Moreover, the good agreement of the pressure predictions over such a wide range of supersonic to strongly hypersonic conditions lends additional a posteriori support to the universality of the incipient separation angle criterion Eq. (31) on which these predictions are based.

IV. Heat Transfer Predictions and Experimental Comparisons

Because Eq. (23) governing H is of the same form as the equation governing T derived in Ref. 7, we may adapt its numerical results to the present problem. Thus, we obtain for the local small-scale interactive heat transfer behavior (normalized to the upstream noninteractive value) the typical predictions shown in Fig. 4 for $Pr = 1$. These curves (which do not include allowance of the large-scale hypersonic viscous interaction effect that occurs downstream of the corner), clearly show how the adverse pressure-induced reduction in heating grows in direct proportion to the strength of the interaction, with a sharply peaked minimum at the shock foot, i.e., at the corner, as long as the flow remains unseparated. For the stronger interactions that promote local separation, on the other hand, this peaking quickly disappears into a broader smoothed minimum. These local heating minima would be slightly lower for $Pr = 0.72$ in air.

For sufficiently weak interactions where separation does not occur, it is shown in Appendix B that the solutions of Eq. (23) upstream of the interaction are virtually independent of the external pressure field and, hence, can be applied to hypersonic as well as supersonic external Mach number flows. Accordingly, we may seek to directly compare the present theoretical predictions with experimental data obtained for hypersonic corner interactions provided we account for the added influence of the larger-scale viscous–inviscid effect caused by the thickening boundary layer downstream of the corner when M_∞ is large (the appropriate theory has been given in detail by

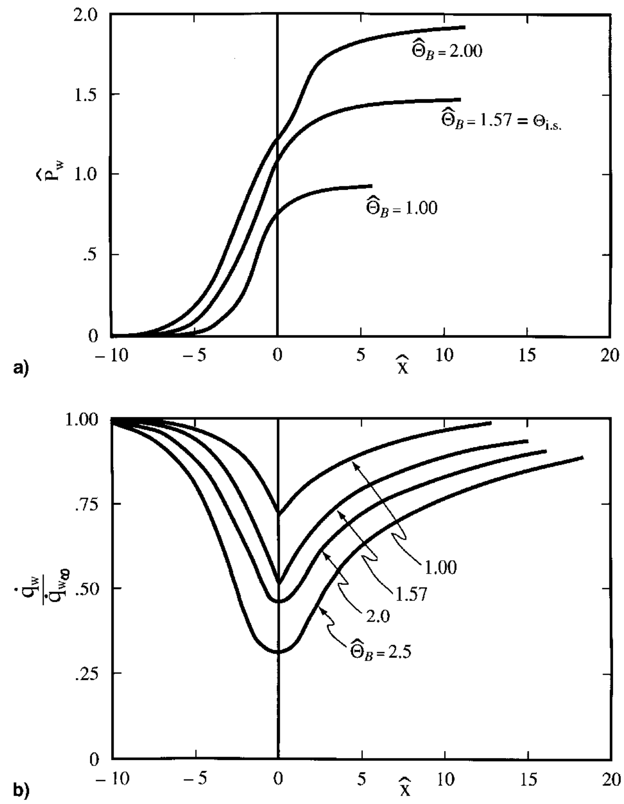

Fig. 4 Typical nondimensional interactive wall property distributions according to asymptotic theory: a) pressure and b) heat transfer.

Stollery[19]). This is done in Fig. 5, where the local heat transfer distributions measured by Needham[20] in a Mach 9.7 interacting corner flow of varying angles are compared with the theoretical predictions. Regarding the small-scale, triple-deck aspect of the interaction upstream and immediately at the corner, it is seen that the predicted deepening of the heat transfer reduction with increasing interactive strength and its associated sharply peaked minimum at the shock foot are both clearly confirmed by the data. Moreover, the rather abrupt disappearance of the sharp minimum peak after separation indicated by the theory (see preceding text) is also experimentally corroborated. Needham[20] suggested that this feature would in fact serve as an incipient separation onset criterion: The present work now provides firm theoretical support for this empirical conclusion. Additional heat transfer data from several other investigators[21,22] also bear out these conclusions, as do numerical results obtained from interacting boundary-layer solutions on a rounded compression ramp.[23]

Concerning the interaction zone downstream of the corner, it is seen from Fig. 5 that the locally developing large-scale viscous interaction effect[19] at this truly hypersonic Mach number of 9.7 (shown here for the single corner angle treated in Ref. 19) completely overwhelms the wake of the small-scale, triple-deck structure, producing a very rapid pressure and heat transfer rise in rough agreement with the experimental data. This behavior is consistent with basic analysis showing that cooled hypersonic boundary layers in adverse pressure gradients actually thin out with a corresponding increase in the heat transfer.[24]

Concluding Remarks

This investigation has established a firm theoretical foundation (in the high Re limit of triple-deck theory) for well-established but up to now, empirical, deflection angle, pressure, and local heat transfer criteria for incipient separation caused by shock-boundary-layer interaction. We have addressed the

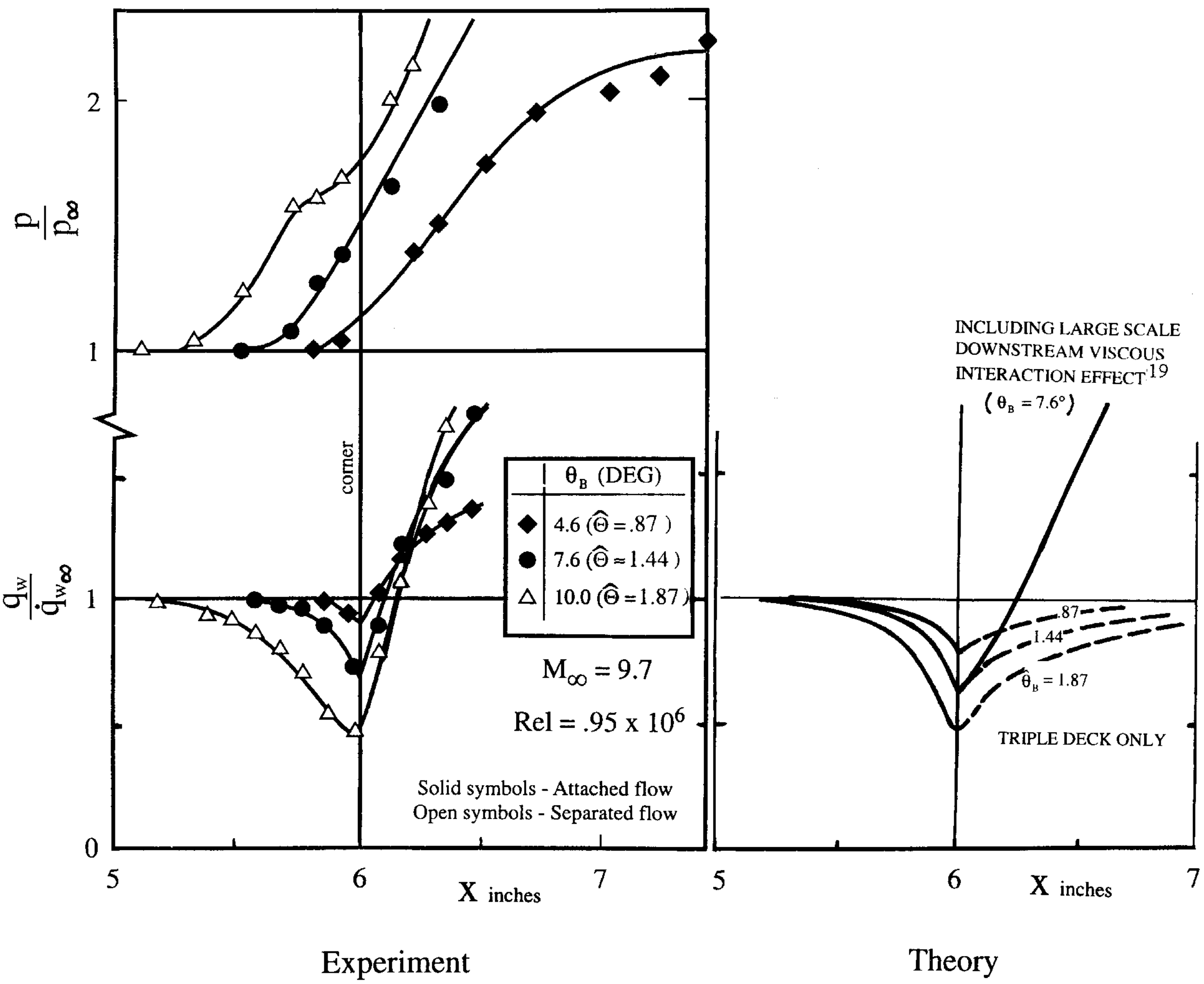

Fig. 5 Comparison of theory with experiment. Validation of Needhams empirical heat transfer criterion for incipient separation.

issue of hypersonic flows, where the inviscid flow pressure–streamline deflection relationship involves the nonlinear effects occurring at high Mach numbers. The analysis has further emphasized the perturbation heat transfer aspect of the interaction zone by adding explicit treatment of the disturbance energy equation.

Appendix A: Parameter S_I

The cold-wall interaction parameter S_I, defined by Eq. (27), can be conveniently rewritten in terms of the combined supersonic–hypersonic interaction parameter $\tilde{\tilde{\chi}} = M_0\tilde{\chi}/\beta$ as

$$S_I = \lambda^{5/4}(\tilde{\tilde{\chi}})^{1/4}\left(\frac{T_t}{T_w}\right)^{\omega+1/2}\cdot\frac{M_\infty}{(T_t/T_\infty)^{1+\omega/2}}\left(\frac{T_t}{T_R}\right)^{1-\omega/2}$$

$$= \left\{\frac{\lambda^{5(1+2\omega)/2}(\tilde{\tilde{\chi}})^{1/2(1+2\omega)}(T_t/T_R)^{(1-\omega)/1+2\omega}[M_\infty^2(T_\infty/T_t)^{2+\omega}]^{1/1+2\omega}}{T_w/T_t}\right\}^{(1+2\omega)/2} \quad \text{(A1)}$$

where Eq. (A1) is proportional to $\tilde{\chi}^{1/4}\ (T_t/T_w)^{\omega+1/2}$, the cold-wall interaction parameter used in the consideration of very strong cold-wall interactions.[14] However, closer inspection and evaluation of the alternative expression [Eq. (A2)] reveals that S_I is, in fact, still small under the typical highly cooled wall temperatures encountered in practice ($T_w/T_t \gtrsim 0.10$), unless $\tilde{\chi}$ takes on unrealistically large values.

Appendix B: Vertical Flowfield

The underlying reason for the deflection-angle incipient separation criterion [Eq. (31)] can be extracted from an analysis of the vertical velocity disturbance field $\hat{v}$, as follows. The key basic equation governing this can be obtained by successively differentiating momentum Eq. (22) with respect to $\hat{y}$ and $\hat{x}$, and then using $dp/dy = 0$ and the continuity equation to largely eliminate $\hat{u}$ in favor of $\hat{v}$; the result is

$$\frac{\partial^4\hat{v}}{\partial\hat{y}^4} - \hat{u}\,\frac{\partial}{\partial\hat{x}}\left(\frac{\partial^2\hat{v}}{\partial\hat{y}^2}\right) + \left(\frac{\partial^2\hat{u}}{\partial\hat{y}^2}\right)\frac{\partial\hat{v}}{\partial\hat{x}} = \hat{v}\,\frac{\partial^3\hat{v}}{\partial\hat{y}^3} - \frac{\partial\hat{v}}{\partial\hat{y}}\left(\frac{\partial^2\hat{v}}{\partial\hat{y}^2}\right) \quad \text{(B1)}$$

It is important to note here that the pressure and, hence, its specific law of deflection angle dependence, has been eliminated explicitly. Now, under the conditions prior to or just at incipient separation, the interactive viscous flow close to the surface can be taken as one of small angular disturbance, i.e., small v/u, regardless of whether it is supersonic or strongly hypersonic; in the leading approximation for such a flow; we thus neglect the second-order terms in v on the right side of Eq. (B1) and to the same consistent order of approximation replace the coefficients of v involving u on the left by their undisturbed values based on $\hat{u} = \hat{y}$. Regardless of the Mach-number regime or whether the pressure-deflection angle relationship is linear or not, the $\hat{v}$ field close to the wall for unseparated to incipiently separated flow is therefore governed by the simplified equation

$$\frac{\partial^4\hat{v}}{\partial\hat{y}^4} - \hat{y}\,\frac{\partial}{\partial\hat{x}}\left(\frac{\partial^2\hat{v}}{\partial\hat{y}^2}\right) \approx 0 \quad \text{(B2)}$$

We note that this equation is a fundamental relationship underlying the so-called *free interaction* concept.

Equation (B2), which is homogenous in $\partial^2\hat{v}/\partial\hat{y}^2$, is to be solved subject to the inner impermeable wall no-slip conditions that $\hat{v}(x, 0) = \partial\hat{v}/\partial\hat{y}(\hat{x}, 0) = 0$ and an outer far-field condition of vanishing viscous shear stress $\partial^2\hat{v}/\partial y^2 \to 0$ at sufficiently large $\hat{y}$. This solution involving an Airy function, $A_i(\hat{y})$, is well known: Confining attention to the region $x < 0$ upstream of the corner or shock impingement point it is[1]

$$\hat{v} = ce^{0.827\hat{x}}\left[z\int_0^{\hat{y}} A_i(z)\,dz - \int_0^{\hat{y}} zA_i(z)\,dz\right] \tag{B3}$$

where $z = (0.827)^{1/3}\,\hat{y}$, and c is a constant that is directly proportional to the corner deflection angle. In particular, at large $\hat{y}$ outside the inner deck, where $\hat{u} \simeq \hat{y}$, Eq. (B3) yields the following value for the local streamline slope (flow angle):

$$\hat{v}/\hat{u} = c'e^{0.827\hat{x}} \tag{B4}$$

where $c' = (0.827)^{1/3}c\int_0^\infty A_i(z)\,dz = 0.313c$. Thus, at the station $\hat{x} = 0$, where incipient separation approximately occurs, Eq. (B4) predicts that the flow deflection angle is proportional to a constant, which in turn, is directly proportional to the nondimensional compression corner angle $\hat{\theta}_{\text{i.s.}}$ regardless of the Mach-number regime. This is exactly what is observed and embodied in Eq. (31).

Regarding the corresponding heat transfer aspects, we note that the inner deck energy equation, Eq. (23), reduces to the following relation for the upstream v field of Eq. (B2):

$$\left(\frac{1}{Pr}\frac{\partial^2}{\partial\hat{y}^2} - \hat{y}\frac{\partial}{\partial\hat{x}}\right)\hat{H}' = \hat{v} \tag{B5}$$

where $\hat{H} \equiv \hat{H} - \hat{y}$ is the local perturbation and $\hat{H}'(x, 0) = 0$. This equation shows that $\hat{H}'$, like $\hat{v}$, is independent of the details of the pressure-deflection relationship or the Mach-number regime in the upstream interaction zone.

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